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FORCE INTERACTION OF A SPHERE AND A VISCOUS LIQUID IN THE PRESENCE OF A WALL

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The problem of the force interaction of a vibrating sphere and a viscous liquid bounded from outside by a rigid wall at rest is studied under the assumption that the largest displacement of the sphere is small compared to its radius and the radius of the sphere is small compared to the distance between the sphere and the wall surface. The liquid flow and the force exerted by the liquid on the sphere are determined.

A number of theoretical problems of the motion of a solid body in an ideal liquid under vibration have been studied [1–7]. The most important result of the work performed was the finding of new effects in the mean motion of inclusions in the liquid. A natural extension of studies in this area is concerned, in particular, with examining the behavior of solid inclusions in a viscous liquid under conditions similar to the ones considered previously.

In [8], we formulated a principle according to which the main reason for the effects of mean motion of inclusions in a liquid under vibration is that the inclusions can move in different directions under dissimilar conditions. The validity of this principle is vividly demonstrated by the problem of the motion of a solid sphere in a liquid in the presence of a wall. For an ideal liquid, a solution of this problem containing the effect of mean motion of inclusions is given in [7] (see also [3, 6, 8]). For a viscous liquid, problems of this kind are very complicated. An important step in the investigation of the effects of mean motion of solid inclusions in a viscous liquid is an examination of the force interaction of a solid inclusion and a viscous liquid with specified motion of the inclusion.

1. We consider the following problem. An absolutely rigid sphere is present in a viscous, incompressible liquid bounded from outside by an absolutely rigid, planar wall surface. The wall is at rest, and the sphere performs specified periodic translational vibrations with period T relative to the rectangular coordinates X_1, X_2, and X_3. The wall surface coincides with the plane $X_1 = 0$. The region occupied by the liquid is contained in the half-space $X_1 \geqslant 0$. The position of the wall is given by the radius-vector $\boldsymbol{Z} = Z\boldsymbol{e}_1$ of the center of the sphere, where Z is a periodic function of t with period T and $\boldsymbol{e}_1 = (1, 0, 0)$. The liquid flow does not depend on the initial conditions. It is required to determine the force interaction between the sphere and the liquid, i.e., the force $\boldsymbol{F}$ exerted by the liquid on the sphere (the force acting on the liquid from the sphere is $-\boldsymbol{F}$).

We assume that $\tau = t/T$, a is the radius of the sphere, $x_1 = X_1/a$, $x_2 = X_2/a$, and $x_3 = X_3/a$, $R = \sqrt{x_1^2 + x_2^2 + x_3^2}$, ρ, $\boldsymbol{V}$, and P are the liquid density, velocity, and pressure, respectively, $\boldsymbol{v} = T\boldsymbol{V}/a$, $p = T^2P/(\rho a^2)$ ν is the kinematic viscosity of the liquid, $\mathrm{Re} = a^2/(\nu T)$ is the Reynolds number, $\mathcal{P}$ is the stress tensor in the liquid, $\boldsymbol{W} = (dZ/dt)\boldsymbol{e}_1$ is the velocity of the sphere, $\hat{W}$ is the largest magnitude of $|\boldsymbol{W}|$, $\boldsymbol{w} = \boldsymbol{W}/\hat{W} = w\boldsymbol{e}_1$ $\Big(w = \mathrm{Real} \sum\limits_{m=1}^{\infty} w_m e^{2m\pi i\tau}\Big)$, $\delta = \hat{W}T/a$, $\langle Z\rangle = \dfrac{1}{T}\displaystyle\int_t^{t+T} Z\,dt$, $\varepsilon = a/\langle Z\rangle$, $z = Z/\langle Z\rangle$, (q) is

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the surface of the sphere described by the equation $(x_1 - z/\varepsilon)^2 + x_2^2 + x_3^2 = 1$, $\boldsymbol{n}$ is the outer unit normal to (q), and (Q) is the surface of the wall described by the equation $x_1 = 0$.

The formula for the force exerted by the liquid on the sphere, the Navier-Stokes and continuity equations, and the conditions that should be satisfied on (q) and (Q) and for $R \to \infty$ have the form

$$\boldsymbol{F} = \iint\limits_{(q)} \mathcal{P} \cdot \boldsymbol{n}\, dq; \tag{1.1}$$

$$\frac{\partial \boldsymbol{v}}{\partial \tau} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p + \frac{1}{\mathrm{Re}} \Delta \boldsymbol{v}, \quad \nabla \cdot \boldsymbol{v} = 0; \tag{1.2}$$

$$\boldsymbol{v} = \delta \boldsymbol{w} \text{ on } (q), \quad \boldsymbol{v} = 0 \text{ on } (Q), \quad \boldsymbol{v} \to 0 \text{ as } R \to \infty. \tag{1.3}$$

2. We assume that the values of δ and ε are small compared to unity and the values of δ are small compared to the values of ε^3. We note that the values of Re are not assumed to be small or large compared to unity.

We find an approximate solution of the problem (1.2), (1.3). The approach considered below extended the method of [9] for solving the problem of potential flow of an ideal liquid to the problem of viscous flow with Reynolds number that are not small or larger compared to unity.

2.1 We consider the problem of liquid flow in the absence of a wall. In the coordinate system $X_1' = X_1 - Z$, $X_2' = X_2$, $X_3' = X_3$ (in which the sphere is immovable), we have

$$\frac{\partial \boldsymbol{v}_0'}{\partial \tau} + (\boldsymbol{v}_0' \cdot \nabla')\boldsymbol{v}_0' = -\nabla' p_0' + \frac{1}{\mathrm{Re}} \Delta' \boldsymbol{v}_0' - \delta \frac{d\boldsymbol{w}}{d\tau}, \quad \nabla' \cdot \boldsymbol{v}_0' = 0; \tag{2.1}$$

$$\boldsymbol{v}_0' = 0 \text{ for } r = 1, \qquad \boldsymbol{v}_0' \to -\delta \boldsymbol{w} \text{ as } r \to \infty, \tag{2.2}$$

where $\boldsymbol{v}_0' = T\boldsymbol{V}_0'/a$ ($\boldsymbol{V}_0'$ is the liquid velocity), $p_0' = T^2(P_0'/(\rho a^2)$ (P_0' is the pressure in the liquid), and $r = \sqrt{x_1'^2 + x_2'^2 + x_3'^2}$ ($x_1' = X_1'/a$, $x_2' = X_2'/a$, and $x_3' = X_3'/a$).

We assume that as $\delta \to 0$,

$$\boldsymbol{v}_0' \sim \delta \boldsymbol{v}_0'^{(1)}, \qquad p_0' \sim \delta p_0'^{(1)}. \tag{2.3}$$

Using (2.1)–(2.3), we obtain

$$\frac{\partial \boldsymbol{v}_0'^{(1)}}{\partial \tau} = -\nabla' p_0'^{(1)} + \frac{1}{\mathrm{Re}} \Delta' \boldsymbol{v}_0'^{(1)} - \frac{d\boldsymbol{w}}{d\tau}, \quad \nabla' \cdot \boldsymbol{v}_0'^{(1)} = 0; \tag{2.4}$$

$$\boldsymbol{v}_0'^{(1)} = 0 \text{ for } r = 1, \qquad \boldsymbol{v}_0'^{(1)} \to -\boldsymbol{w} \text{ as } r \to \infty. \tag{2.5}$$

The problem (2.4), (2.5) has the solution

$$v_{0r}'^{(1)} = \frac{1}{r^2 \sin\theta} \frac{\partial \psi_0}{\partial r}, \qquad v_{0\theta}'^{(1)} = -\frac{1}{r \sin\theta} \frac{\partial \psi_0}{\partial r}; \tag{2.6}$$

$$p_0'^{(1)} = \Big\{\Big[-\frac{\partial^2}{\partial \tau\, \partial r} + \frac{1}{\mathrm{Re}}\Big(\frac{\partial^3}{\partial r^3} - \frac{2}{r^2}\frac{\partial}{\partial r} + \frac{4}{r^3}\Big)\Big]\psi_0 - \frac{dw}{d\tau} r \sin^2\theta\Big\} \frac{\cos\theta}{\sin^2\theta} + c_0, \tag{2.7}$$

where

$$\psi_0 = \frac{1}{2}\Big\{-wr^2 + \mathrm{Real} \sum_{m=1}^{\infty} \frac{w_m}{q_m}\Big[\frac{q_m^2 + 3q_m + 3}{q_m r} - \frac{3r^{1/2} K_{3/2}(q_m r)}{K_{1/2}(q_m)}\Big] e^{2m\pi i \tau}\Big\} \sin^2\theta$$

[$q_m = (1+i)\sqrt{m\pi \mathrm{Re}}$ and $K_{1/2}$ and $K_{3/2}$ are the Macdonald functions], θ is the angle between the vectors $(1, 0, 0)$ and (x_1', x_2', x_3'), $v_{0r}'^{(1)}$ and $v_{0\theta}'^{(1)}$ are the r- and θ-components of the vector $\boldsymbol{v}_0'^{(1)}$, and c_0 is a function of τ.

2.2. We convert to the coordinates X_1, X_2, and X_3. We determine the error that arises in the plane $x_1 = 0$ when $\boldsymbol{v}$ in the condition $\boldsymbol{v} = 0$ for $x_1 = 0$ is replaced by $\delta(\boldsymbol{v}_0'^{(1)} + \boldsymbol{w})$. According to (2.6), we have

$$\delta(\boldsymbol{v}_0'^{(1)} + \boldsymbol{w})\Big|_{x_1=0} = \delta\Big[(\nabla\chi)\Big|_{x_1=0} + \boldsymbol{\xi}\Big], \tag{2.8}$$

where

$$\chi = \frac{1}{2}\varepsilon^2 A \frac{z - \varepsilon x_1}{[(z-\varepsilon x_1)^2 + \varepsilon^2(x_2^2 + x_3^2)]^{3/2}} \qquad \Big(A = \text{Real}\sum_{m=1}^{\infty} w_m \frac{q_m^2 + 3q_m + 3}{q_m^2} e^{2m\pi i\tau}\Big)$$

and $\boldsymbol{\xi}$ is a small quantity compared to ε^α (α is any positive number).

We consider the following problem of liquid flow in the absence of a sphere, whose solution compensates for the error (2.8):

$$\frac{\partial \boldsymbol{v}_c}{\partial \tau} + (\boldsymbol{v}_c \cdot \nabla)\boldsymbol{v}_c = -\nabla p_c + \frac{1}{\text{Re}}\Delta \boldsymbol{v}_c, \qquad \nabla\cdot \boldsymbol{v}_c = 0; \tag{2.9}$$

$$\boldsymbol{v}_c = -\delta(\nabla\chi + \boldsymbol{\xi}) \text{ for } x_1 = 0, \qquad \boldsymbol{v}_c \to 0 \text{ as } R \to \infty, \tag{2.10}$$

where $\boldsymbol{v}_c = T\boldsymbol{V}_c/a$ ($\boldsymbol{V}_c$ is the liquid velocity) and $p_c = T^2 P_c/(\rho a^2)$ (P_c is the liquid pressure).

We assume that as $\delta \to 0$,

$$\boldsymbol{v}_c \sim \delta \boldsymbol{v}_c^{(1)}, \qquad p_c \sim \delta p_c^{(1)}. \tag{2.11}$$

Using (2.9)–(2.11), we obtain

$$\frac{\partial \boldsymbol{v}_c^{(1)}}{\partial \tau} = -\nabla p_c^{(1)} + \frac{1}{\text{Re}}\Delta \boldsymbol{v}_c^{(1)}, \qquad \nabla\cdot \boldsymbol{v}_c^{(1)} = 0; \tag{2.12}$$

$$\boldsymbol{v}_c^{(1)} = -\nabla\chi - \boldsymbol{\xi} \text{ for } x_1 = 0, \qquad \boldsymbol{v}_c^{(1)} \to 0 \text{ as } R \to \infty. \tag{2.13}$$

In (2.12), (2.13), we make the substitution

$$\boldsymbol{v}_c^{(1)} = \boldsymbol{v}^* + \nabla\chi^*, \tag{2.14}$$

where

$$\chi^* = \frac{1}{2}\varepsilon^2 A \frac{z + \varepsilon x_1}{[(z+\varepsilon x_1)^2 + \varepsilon^2(x_2^2 + x_3^2)]^{3/2}}.$$

As a result, we obtain

$$\frac{\partial \boldsymbol{v}^*}{\partial \tau} = -\nabla\Big(p_c^{(1)} + \frac{\partial \chi^*}{\partial \tau}\Big) + \frac{1}{\text{Re}}\Delta \boldsymbol{v}^*, \qquad \nabla\cdot \boldsymbol{v}^* = 0; \tag{2.15}$$

$$\boldsymbol{v}^* = -\nabla(\chi + \chi^*) - \boldsymbol{\xi} \text{ for } x_1 = 0, \qquad \boldsymbol{v}^* \to 0 \text{ as } R \to \infty. \tag{2.16}$$

The problem (2.15), (2.1) has the following solution, which satisfies the first condition of (2.15), the second condition of (2.15), and the first condition of (2.16) with accuracy up to ε^4, ε^3, and ε^α, respectively, and rigorously satisfies the second condition of (2.16):

$$v_1^* = 0,$$
$$v_L^* = \frac{3\varepsilon^4 x_L}{[z^2 + \varepsilon^2(x_2^2 + x_3^2)]^{5/2}} \text{Real}\sum_{m=1}^{\infty} w_m \frac{q_m^2 + 3q_m + 3}{q_m^2} e^{-q_m x_1 + 2m\pi i\tau} \quad (L = 2,\, 3); \tag{2.17}$$

$$p_c^{(1)} + \frac{\partial \chi^*}{\partial \tau} = C_c, \tag{2.18}$$

where v_1^*, v_2^*, and v_3^* are the x_1-, x_2-, and x_3- components of the vector $\boldsymbol{v}^*$ and C_c is a function of τ. In view of this, the problem (2.90, (2.10) has solution (2.14), (2.17), (2.18).

The expression for $\delta(\boldsymbol{v}_0'^{(1)} + \boldsymbol{w} + \boldsymbol{v}_c^{(1)})$ given by relations (2.6), (2.14), and (2.17) satisfies the condition $\delta(\boldsymbol{v}_0'^{(1)} + \boldsymbol{w} + \boldsymbol{v}_c^{(1)}) = 0$ for $x_1 = 0$ with accuracy up to $\delta\varepsilon^\alpha$.

2.3. We convert to the coordinates X_1', X_2', and X_3'. We determine the error that arises on the sphere $r = 1$ when $\boldsymbol{v} - \delta \boldsymbol{w}$ in the condition $\boldsymbol{v} - \delta \boldsymbol{w} = 0$ for $r = 1$ is replaced by $\delta(\boldsymbol{v}_0'^{(1)} + \boldsymbol{v}_c^{(1)})$. According to (2.6), (2.14), and (2.17), we have

$$\delta(\boldsymbol{v}_0'^{(1)} + \boldsymbol{v}_c^{(1)})\Big|_{r=1} = \delta\Big[(\nabla\chi^*)\Big|_{r=1} + \boldsymbol{\xi}'\Big], \tag{2.19}$$

where $\boldsymbol{\xi}'$ is a small quantity compared to $\varepsilon^{\alpha'}$(α' is any positive number).

We consider the following problem of liquid flow in the absence of a wall, whose solution compensates for the error (2.19):

$$\frac{\partial \boldsymbol{v}_c'}{\partial \tau} + (\boldsymbol{v}_c' \cdot \nabla')\boldsymbol{v}_c' = -\nabla' p_c' + \frac{1}{\mathrm{Re}}\,\Delta' \boldsymbol{v}_c' - \delta\,\frac{d\boldsymbol{w}}{d\tau}, \qquad \nabla' \cdot \boldsymbol{v}_c' = 0; \tag{2.20}$$

$$\boldsymbol{v}_c' = -\delta(\nabla\chi^* + \boldsymbol{\xi}') \ \text{ for } \ r = 1, \qquad \boldsymbol{v}_c' \to 0 \ \text{ as } \ r \to \infty, \tag{2.21}$$

where $\boldsymbol{v}_c' = T\boldsymbol{V}_c'/a$ ($\boldsymbol{V}_c'$ is the liquid velocity) and $p_c' = T^2 P_c'/(\rho a^2)$ (P_c' is the liquid pressure).

We assume that as $\delta \to 0$.

$$\boldsymbol{v}_c' \sim \delta \boldsymbol{v}_c'^{(1)}, \qquad p_c' \sim \delta p_c'^{(1)}. \tag{2.22}$$

Using (2.20)–(2.22), we obtain

$$\frac{\partial \boldsymbol{v}_c'^{(1)}}{\partial \tau} = -\nabla' p_c'^{(1)} + \frac{1}{\mathrm{Re}}\,\Delta' \boldsymbol{v}_c'^{(1)} - \frac{d\boldsymbol{w}}{d\tau}, \qquad \nabla' \cdot \boldsymbol{v}_c'^{(1)} = 0; \tag{2.23}$$

$$\boldsymbol{v}_c'^{(1)} = -\nabla\chi^* - \boldsymbol{\xi}' \ \text{ for } \ r = 1, \qquad \boldsymbol{v}_c'^{(1)} \to 0 \ \text{ as } \ r \to \infty. \tag{2.24}$$

The problem (2.23), (2.24) has the solution

$$v_{cr}'^{(1)} = \frac{1}{r^2 \sin\theta}\,\frac{\partial \psi_c}{\partial \theta}, \qquad v_{c\theta}'^{(1)} = -\frac{1}{r \sin\theta}\,\frac{\partial \psi_c}{\partial r}; \tag{2.25}$$

$$p_c'^{(1)} = \Big\{\Big[-\frac{\partial^2}{\partial\tau\,\partial r} + \frac{1}{\mathrm{Re}}\Big(\frac{\partial^3}{\partial r^3} - \frac{2}{r^2}\,\frac{\partial}{\partial r} + \frac{4}{r^3}\Big)\Big]\psi_c - \frac{dw}{d\tau}\, r \sin^2\theta\Big\}\frac{\cos\theta}{\sin^2\theta} + c_c, \tag{2.26}$$

which satisfies the first condition of (2.24) with accuracy to ε^3 and exactly satisfies (2.23) and the second condition of (2.4). Here

$$\psi_c = \frac{1}{16}\,\varepsilon^3\,\mathrm{Real}\sum_{m=1}^{\infty} w_m\,\frac{q_m^2 + 3q_m + 3}{q_m^3}\Big[\frac{q_m^2 + 3q_m + 3}{q_m r} - \frac{3r^{1/2} K_{3/2}(q_m r)}{K_{1/2}(q_m)}\Big]e^{2m\pi i\tau}\sin^2\theta$$

and c_c is a function of τ.

The expression for $\delta(\boldsymbol{v}_0'^{(1)} + \boldsymbol{v}_c^{(1)} + \boldsymbol{v}_c'^{(1)})$ given by formulas (2.6), (2.14), (2.17), and (2.25) satisfies the condition $\delta(\boldsymbol{v}_0'^{(1)} + \boldsymbol{v}_c^{(1)} + \boldsymbol{v}_c'^{(1)}) = 0$ for $r = 1$ with accuracy to $\delta\varepsilon^3$.

2.4. According to the aforesaid, the problem (1.2), (1.3) has an approximate solution given by the formulas

$$\boldsymbol{v} = \delta(\boldsymbol{v}_0'^{(1)} + \boldsymbol{w} + \boldsymbol{v}_c^{(1)} + \boldsymbol{v}_c'^{(1)}), \qquad p = \delta\Big(p_0'^{(1)} + p_c^{(1)} + p_c'^{(1)} + \frac{dw}{d\tau}\,x_1\Big) + c, \tag{2.27}$$

and (2.6), (2.7), (2.14), (2.17), (2.18), (2.25), and (2.26) (c is a function of τ). This solution satisfies (1.2) and the first two conditions of (1.3) with accuracy to $\delta\varepsilon^3$ and exactly satisfies the last condition of (1.3).

3. Using (1.1), (2.6), (2.7), (2.14), (2.17), (2.18), and (2.25)–(2.27), we obtain

$$\boldsymbol{F} = -\frac{2\pi a^3 \rho \hat{W}}{3T}\Big\{\frac{dw}{d\tau} - 18\pi\,\mathrm{Imag}\sum_{m=1}^{\infty} m w_m \frac{q_m + 1}{q_m^2}\,e^{2m\pi i\tau}$$

$$+ \frac{3}{8}\,\varepsilon^3\Big[\frac{dw}{d\tau} - 12\pi\,\mathrm{Imag}\sum_{m=1}^{\infty} m w_m \frac{q_m + 1}{q_m^4}\Big(q_m^2 + \frac{3}{2}\,q_m + \frac{3}{2}\Big)e^{2m\pi i\tau}\Big]\Big\}\boldsymbol{e}_1. \tag{3.1}$$

Formula (3.1) gives the force interaction between the sphere and the liquid.

From (3.1) it follows that for $\varepsilon = 0$,

$$\boldsymbol{F} = -\frac{2\pi a^3 \rho \hat{W}}{3T}\left[\frac{dw}{d\tau} - 18\pi \operatorname{Imag}\sum_{m=1}^{\infty} m w_m \frac{q_m + 1}{q_m^2} e^{2m\pi i\tau}\right]\boldsymbol{e}_1. \quad (3.2)$$

Formula (3.2) is in agreement with the formula of [10] for the force acting on a solid sphere from a viscous liquid that is unbounded from outside.

4. The above approach to determining viscous flow with an inclusion in the presence of a wall is suitable for studying the effects of mean motion of inclusions in a viscous liquid. It can be used, in particular, to investigate the effect of the viscosity of a liquid on the paradoxical equilibrium of a solid inclusion present in the liquid and the "levitation" of the inclusion under vibrating actions on the liquid [1, 3, 7].

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